Quotient Law:

Consider ~~a~~ a quantity A and we do not know whether A is tensor or not. If the inner product of A with an arbitrary tensor is itself a tensor then A is also a tensor. This is called the quotient law.

Let $A(i, j, k) \rightarrow$ components of a tensor

Let inner product of ~~(A,~~ $A(i,j,k)$) with an arbitrary tensor B^{pq} is a contravariant tensor of first rank

$$A(i,j,k)\, B^{jk} = C^{i} \qquad \text{—— (1)}$$

j,k (Repeated index) are summed over.

Let $\bar{A}(\alpha, \beta, \gamma) \rightarrow N^3$ functions in the barred coordinates and satisfy

$$\bar{A}(\alpha,\beta,\gamma)\, \bar{B}^{\beta\gamma} = \bar{C}^{\alpha} \qquad \text{—— (2)}$$

to write $\bar{B}$ and $\bar{C}$ in terms of unbarred components, we write the above eq. as

$$\bar{A}(\alpha,\beta,\gamma)\,\frac{\partial \bar{x}^\beta}{\partial x^j}\frac{\partial \bar{x}^\gamma}{\partial x^k}B^{jk} = \frac{\partial \bar{x}^\alpha}{\partial x^i}C^i$$

$$= \frac{\partial \bar{x}^\alpha}{\partial x^i}A(i,j,k)\,B^{jk}$$

or $$\left[\bar{A}(\alpha,\beta,\gamma)\,\frac{\partial \bar{x}^\beta}{\partial x^j}\frac{\partial \bar{x}^\gamma}{\partial x^k} - \frac{\partial \bar{x}^\alpha}{\partial x^i}A(i,j,k)\right]B^{jk} = 0 \quad \text{(from eq. 1)} \qquad (3)$$

This is true for ~~or~~ an arbitrary tensor B^{jk}. Therefore,

$$\bar{A}(\alpha,\beta,\gamma)\,\frac{\partial \bar{x}^\beta}{\partial x^j}\frac{\partial \bar{x}^\gamma}{\partial x^k} = \frac{\partial \bar{x}^\alpha}{\partial x^i}A(i,j,k)$$

Now multiplying $\frac{\partial x^j}{\partial \bar{x}^\rho}\frac{\partial x^k}{\partial \bar{x}^\sigma}$ (inner multiplication) on both sides, we obtain

$$\bar{A}(\alpha,\beta,\gamma)\,\delta_{\beta\rho}\,\delta_{\gamma\sigma} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^\rho}\frac{\partial x^k}{\partial \bar{x}^\sigma}A(i,j,k)$$

$$\boxed{\bar{A}(\alpha,\rho,\sigma) = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^\rho}\frac{\partial x^k}{\partial \bar{x}^\sigma}A(i,j,k)} \qquad (4)$$

From eq. (4), we see that $A(i,j,k)$ is a tensor of contravariant rank one and covariant rank 2. Now writing Eq. (4) in tensor notation

$$\bar{A}^\alpha_{\rho\sigma} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^\rho}\frac{\partial x^k}{\partial \bar{x}^\sigma}A^i_{jk} \qquad (5)$$

<u>Conjugate or reciprocal tensors</u>

Let A_{ij} → Symmetric covariant tensor of rank 2, such that when A_{ij} is expressed as a matrix then $\det(A_{ij}) = |A_{ij}| \neq 0$. Now define

$$B^{ij} = \frac{\text{cofactor of } A_{ij}}{|A_{ij}|}$$

Then B^{ij} is a symmetric contravariant tensor of rank 2 called as the conjugate tensor of A_{ij} and

$$\boxed{B^{ij}A_{kj} = \delta^i_k}$$